$y = \ln x$. What is $y' =$?

$y = \log_b x$

$\Updownarrow$

$b^y = x$

If $y = \ln x$, then $e^y = x$.

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} \cdot f'(x)$$

$$e^y \cdot y' = 1$$

$$\hookrightarrow y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}[\ln x] = \frac{1}{x}}$$

What is $\frac{d}{dx}[\log_b x]$?

$$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$$

Change of Base

$$y = \log_b x = \frac{\ln x}{\ln b}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b} \cdot \overbrace{\frac{d}{dx}[\ln x]}^{\frac{1}{x}} = \frac{1}{x \ln b}$$

$$\boxed{\frac{d}{dx}\left[\log_b x\right] = \frac{1}{x \ln b}}$$

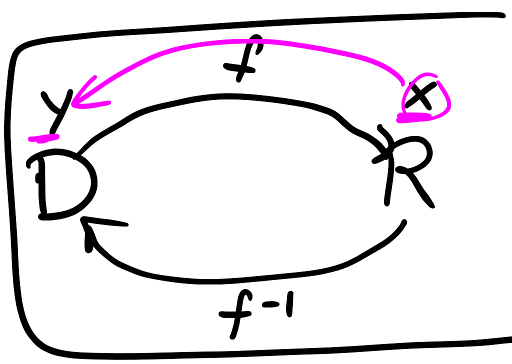
Ex: ① $\dfrac{d}{dx}\left[\log_5(x^2)\right] = \dfrac{1}{x^2 \ln 5} \cdot 2x = \dfrac{2}{x \ln 5}$

② $\dfrac{d}{dx}\left[\ln(\cos x)\right] = \dfrac{1}{\cos x}(-\sin x) = -\tan x$

If f is an invertible function, what is

$$\frac{d}{dx}\left[f^{-1}(x)\right] \ ?$$

Let $y = f^{-1}(x) \longrightarrow f(y) = x$



$$\frac{d}{dx}\left[f(y)\right] = \frac{d}{dx}[x]$$

$$f'(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Theorem: If f is invertible and differentiable, then

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'(f^{-1}(x))}}$$

Ex: Find y'.

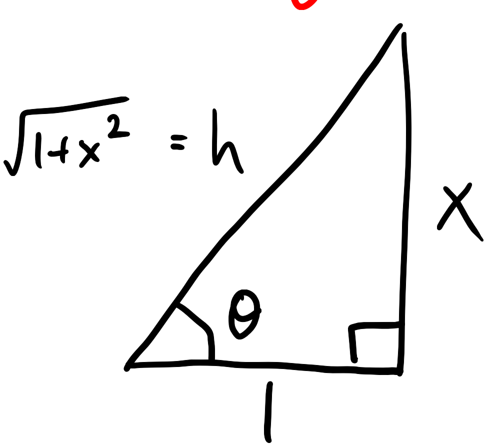
① $y=\underbrace{\arctan}_{f^{-1}(x)}(x)$ $\cancel{\left(=\tan^{-1}x\right)}$

$$f^{-1}(x)=\arctan x \longrightarrow f(x)=\tan x$$

$$f'(x)=\sec^2 x$$

$$f'(f^{-1}(x))=\sec^2(\arctan x)=\left(\sec(\arctan x)\right)^2$$

$$\sec(\underbrace{\arctan x}_{\theta})=\sqrt{1+x^2}$$



$$\sec\theta=\frac{hyp}{adj}=\frac{\sqrt{1+x^2}}{1}=\sqrt{1+x^2}$$

$$1^2+x^2=h^2$$

$$\theta=\arctan x \longrightarrow x=\tan\theta$$

$$f'(f^{-1}(x)) = \left(\sqrt{1+x^2}\right)^2 = 1+x^2$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$